Time-Fractional KdV Equation: Formulation and Solution using Variational Methods

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Abstract

The Lagrangian of the time fractional KdV equation is derived in similar form to the Lagrangian of the regular KdV equation. The variation of the functional of this Lagrangian leads to the Euler-Lagrange equation that leads to the time fractional KdV equation. The Riemann-Liouvulle definition of the fractional derivative is used to describe the time fractional operator in the functional of the Euler-Lagrange formula. The time-fractional term of the derived KdV equation is represented as a Riesz fractional derivative. The variational-iteration method given by He [31] is used to solve the derived time-fractional KdV equation. The calculations of the solution with initial condition $A_0 \operatorname{sech}^2(cx)$ are carried out and demonstrated in 3-dimensions and 2-dimensions figures. We give the comparison and estimates of the role of fractional derivative to the nonlinear and dispersion terms in the fractional KdV equation for unit amplitude $A_0 = 1$. It worth mentioned that, exploitation of the fractional calculus for fractional equation provides not only new types of mathematical construction, but also new physical features of the described phenomena.

Keywords **:** Euler-Lagrange equation, Riemann-Liouvulle definition of the fractional derivative, Riesz fractional derivative, fractional KdV equation, He's variational-iteration method.

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1 Introduction

Because most classical processes observed in the physical world are nonconservative, it is important to be able to apply the power of variational methods to such cases. A method [1] used a Lagrangian that leads to an Euler-Lagrange equation that is, in some sense, equivalent to the desired equation of motion. Hamilton's equations are derived from the Lagrangian and are equivalent to the Euler-Lagrange equation. If a Lagrangian is constructed using noninteger-order derivatives, then the resulting equation of motion can be nonconservative. It

was shown that such fractional derivatives in the Lagrangian describe nonconservative forces [2, 3]. Further study of the fractional Euler-Lagrange can be found in the work of Agrawal [4-6], Baleanu and coworkers [7-9] and Tarasov and Zaslavsky [10, 11].

Recently, fractional calculus has been applied to almost every field of science, engineering and mathematics. The awareness of the importance of this type of equation has grown continuously in last decade include for visco-elasticity and rheology, image processing, mechanics, mechatronics, physics, and control theory, see for instance [12].

On the other hand, the Korteweg–de Vries (KdV) equation has been found to be involved in a wide range of physics phenomena as a model for the evolution and interaction of nonlinear waves. It was first derived as an evolution equation that governing a one dimensional, small amplitude, long surface gravity waves propagating in a shallow channel of water [13]. Subsequently the KdV equation has arisen in a number of other physical contexts as collision-free hydro-magnetic waves, stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics, etc [14]. Certain theoretical physics phenomena in the quantum mechanics domain are explained by means of a KdV model. It is used in fluid dynamics, aerodynamics, and continuum mechanics as a model for shock wave formation, solitons, turbulence, boundary layer behavior, and mass transport. All of the physical phenomena may be considered as nonconservative, so they can be described using fractional differential equations. Therefore, in this paper, our motive is to formulate a time-fractional KdV equation version using the Euler-Lagrange equation via what is called variational method [4-6, 15].

Several methods have been used to solve fractional differential equations such as: the Laplace transformation method [16, 17], the Fourier transformation method [16, 17], the iteration method [18] and the operational method [19]. However, most of these methods are suitable for special types of fractional differential equations, mainly the linear with constant coefficients. Recently, there are some papers deal with the existence and multiplicity of solution of nonlinear fractional differential equation by the use of techniques of nonlinear analysis (fixed-point theorems, Leray–Shauder theory, Adomian decomposition method, variational-iteration method, etc.), see [20-24]. In this paper, the resultant fractional KdV equation will be solved using a variational-iteration method (VIM) [25-27]. In addition, we give the comparison and estimates of the role of fractional derivative to the nonlinear and dispersion terms in the fractional KdV equation for unit amplitude .

This paper is organized as follows: Section 2 is devoted to describe the formulation of the time-fractional KdV (FKdV) equation using the variational Euler-Lagrange method. In section 3, the resultant time-FKdV equation is solved approximately using VIM. Section 4 contains the results of calculations and discussion of these results.

2 The time-fractional KdV equation

The regular KdV equation in (1+1) dimensions is given by [13]

$$\frac{\partial}{\partial t}u(x,t) + A\ u(x,t)\frac{\partial}{\partial x}u(x,t) + B\ \frac{\partial^3}{\partial x^3}u(x,t) = 0, \tag{1}$$

where $u(x,t)$ is a field variable, $x \in R$ is a space coordinate in the propagation direction of the field and $t \in T\ (= [0, T_0])$ is the time variable and A and B are known coefficients.

Using a potential function $v(x,t)$, where $u(x,t) = v_x(x,t)$, gives the potential equation of the regular KdV equation (1) in the form

$$v_{xt}(x,t) + A\ v_x(x,t)v_{xx}(x,t) + B\ v_{xxxx}(x,t) = 0, \tag{2}$$

where the subscripts denote the partial differentiation of the function with respect to the parameter. The Lagrangian of this regular KdV equation (1) can be defined using the semi-inverse method [28, 29] as follows.

The functional of the potential equation (2) can be represented by

$$J(v) = \int\limits_R dx \int\limits_T dt\{v(x,t)[c_1 v_{xt}(x,t) + c_2 A v_x(x,t)v_{xx}(x,t) + c_3 B v_{xxxx}(x,t)]\}, \tag{3}$$

where c_1, c_2 and c_3 are constants to be determined. Integrating by parts and taking $v_t|_R = v_x|_R = v_x|_T = 0$ lead to

$$J(v) = \int\limits_R dx \int\limits_T dt\{v(x,t)[-c_1 v_x(x,t)v_t(x,t) - \frac{1}{2}c_2 A v_x^3(x,t) + c_3 B v_{xx}^2(x,t)]\}. \tag{4}$$

The unknown constants c_i $(i = 1,\ 2,\ 3)$ can be determined by taking the variation of the functional (4) to make it optimal. Taking the variation of this functional, integrating each term by parts and make the variation optimum give the following relation

$$2c_1 v_{xt}(x,t) + 3c_2 A v_x(x,t)v_{xx}(x,t) + 2c_3 B v_{xxxx}(x,t) = 0. \tag{5}$$

As this equation must be equal to equation (2), the unknown constants are given as

$$c_1 = 1/2,\ c_2 = 1/3 \text{ and } c_3 = 1/2. \tag{6}$$

Therefore, the functional given by (6) gives the Lagrangian of the regular KdV equation as

$$L(v_t,\ v_x, v_{xx}) = -\frac{1}{2}v_x(x,t)v_t(x,t) - \frac{1}{6}A v_x^3(x,t) + \frac{1}{2}B v_{xx}^2(x,t). \tag{7}$$

Similar to this form, the Lagrangian of the time-fractional version of the KdV equation can be written in the form

$$\begin{aligned} F({}_0D_t^{\alpha}v,\ v_x, v_{xx}) &= -\frac{1}{2}[{}_0D_t^{\alpha}v(x,t)]v_x(x,t) - \frac{1}{6}Av_x^3(x,t) + \frac{1}{2}Bv_{xx}^2(x,t), \\ 0 &\leq \alpha < 1, \end{aligned} \tag{8}$$

where the fractional derivative is represented, using the left Riemann-Liouville fractional derivative definition as [18, 19]

$${}_aD_t^{\alpha}f(t) = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}[\int_a^t d\tau(t-\tau)^{k-\alpha-1}f(\tau)],\ k-1 \leq \alpha \leq k,\ t \in [a,b]. \tag{9}$$

The functional of the time-FKdV equation can be represented in the form

$$J(v) = \int\limits_R dx \int\limits_T dt\ F({}_0D_t^{\alpha}v,\ v_x, v_{xx}), \tag{10}$$

where the time-fractional Lagrangian $F({}_0D_t^{\alpha}v,\ v_x, v_{xx})$ is defined by (8).

Following Agrawal's method [3, 4], the variation of functional (10) with respect to $v(x,t)$ leads to

$$\delta J(v) = \int\limits_R dx \int\limits_T dt\ \{\frac{\partial F}{\partial_0 D_t^{\alpha}v}\delta_0 D_t^{\alpha}v + \frac{\partial F}{\partial v_x}\delta v_x + \frac{\partial F}{\partial v_{xx}}\delta v_{xx}\}. \tag{11}$$

The formula for fractional integration by parts reads [3, 18, 19]

$$\int_a^b dt f(t)_a D_t^{\alpha}g(t) = \int_a^t dt g(t)_t D_b^{\alpha}f(t), \quad f(t),\ g(t)\ \in [a,\ b]. \tag{12}$$

where ${}_tD_b^{\alpha}$, the right Riemann-Liouville fractional derivative, is defined by [18, 19]

$${}_tD_b^{\alpha}f(t) = \frac{(-1)^k}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}[\int_t^b d\tau(\tau-t)^{k-\alpha-1}f(\tau)],\ k-1 \leq \alpha \leq k,\ t \in [a,b]. \tag{13}$$

Integrating the right-hand side of (11) by parts using formula (12) leads to

$$\delta J(v) = \int\limits_R dx \int\limits_T dt\ [{}_tD_{T_0}^{\alpha}(\frac{\partial F}{\partial_0 D_t^{\alpha}v}) - \frac{\partial}{\partial x}(\frac{\partial F}{\partial v_x}) + \frac{\partial^2}{\partial x^2}(\frac{\partial F}{\partial v_{xx}})]\delta v, \tag{14}$$

where it is assumed that $\delta v|_T = \delta v|_R = \delta v_x|_R = 0$.

Optimizing this variation of the functional $J(v)$, i. e; $\delta J(v) = 0$, gives the Euler-Lagrange equation for the time-FKdV equation in the form

$$ {}_tD^{\alpha}_{T_0}(\frac{\partial F}{\partial_0 D^{\alpha}_t v}) - \frac{\partial}{\partial x}(\frac{\partial F}{\partial v_x}) + \frac{\partial^2}{\partial x^2}(\frac{\partial F}{\partial v_{xx}}) = 0. \quad (15) $$

Substituting the Lagrangian of the time-FKdV equation (8) into this Euler-Lagrange formula (15) gives

$$ -\frac{1}{2}\,{}_tD^{\alpha}_{T_0} v_x(x,t) + \frac{1}{2}\,{}_0D^{\alpha}_t v_x(x,t) + A v_x(x,t) v_{xx}(x,t) + B v_{xxxx}(x,t) = 0. \quad (16) $$

Substituting for the potential function, $v_x(x,t) = u(x,t)$, gives the time-FKdV equation for the state function $u(x,t)$ in the form

$$ \frac{1}{2}[{}_0D^{\alpha}_t u(x,t) -{}_t D^{\alpha}_{T_0} u(x,t)] + A\ u(x,t)\ u_x(x,t) + B\ u_{xxx}(x,t) = 0, \quad (17) $$

where the fractional derivatives ${}_0D^{\alpha}_t$ and ${}_tD^{\alpha}_{T_0}$ are, respectively the left and right Riemann-Liouville fractional derivatives and are defined by (9) and (13).

The time-FKdV equation represented in (17) can be rewritten by the formula

$$ \frac{1}{2}\,{}^R_0D^{\alpha}_t u(x,t) + A\ u(x,t)\ u_x(x,t) + B\ u_{xxx}(x,t) = 0, \quad (18) $$

where the fractional operator ${}^R_0D^{\alpha}_t$ is called Riesz fractional derivative and can be represented by [4, 18, 19]

$$ \begin{aligned} {}^R_0D^{\alpha}_t f(t) &= \frac{1}{2}[{}_0D^{\alpha}_t f(t) +\ (-1)^k{}_tD^{\alpha}_{T_0} f(t)] \\ &= \frac{1}{2}\frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}[\int_a^t d\tau |t-\tau|^{k-\alpha-1} f(\tau)], \\ k-1 &\leq \alpha \leq k,\ t \in [a,b]. \end{aligned} \quad (19) $$

The nonlinear fractional differential equations have been solved using different techniques [18-23]. In this paper, a variational-iteration method (VIM) [24, 25] has been used to solve the time-FKdV equation that formulated using Euler-Lagrange variational technique.

3 Variational-Iteration Method

Variational-iteration method (VIM) [25-27] has been used successfully to solve different types of integer nonlinear differential equations [31-35]. Also, VIM is used to solve linear and nonlinear fractional differential equations [25, 36-38]. This VIM has been used in this paper to solve the formulated time-FKdV equation.

A general Lagrange multiplier method is constructed to solve non-linear problems, which was first proposed to solve problems in quantum mechanics

[25]. The VIM is a modification of this Lagrange multiplier method [26, 27]. The basic features of the VIM are as follows. The solution of a linear mathematical problem or the initial (boundary) condition of the nonlinear problem is used as initial approximation or trail function. A more highly precise approximation can be obtained using iteration correction functional. Considering a nonlinear partial differential equation consists of a linear part $\widehat{L}U(x,t)$, nonlinear part $\widehat{N}U(x,t)$ and a free term $f(x,t)$ represented as

$$\widehat{L}U(x,t) + \widehat{N}U(x,t) = f(x,t), \tag{20}$$

where $\widehat{L}$ is the linear operator and $\widehat{N}$ is the nonlinear operator. According to the VIM, the $(n+1)$th approximation solution of (20) can be given by the iteration correction functional as [24, 25]

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t d\tau \lambda(\tau)[\widehat{L}U_n(x,\tau) + \widehat{N}\widetilde{U}_n(x,\tau) - f(x,\tau)], \; n \geq 0, \tag{21}$$

where $\lambda(\tau)$ is a Lagrangian multiplier and $\widetilde{U}_n(x,\tau)$ is considered as a restricted variation function, i. e; $\delta\widetilde{U}_n(x,\tau) = 0$. Extreme the variation of the correction functional (21) leads to the Lagrangian multiplier $\lambda(\tau)$. The initial iteration can be used as the solution of the linear part of (20) or the initial value $U(x,0)$. As n tends to infinity, the iteration leads to the exact solution of (20), i. e;

$$U(x,t) = \lim_{n\to\infty} U_n(x,t). \tag{22}$$

For linear problems, the exact solution can be given using this method in only one step where its Lagrangian multiplier can be exactly identified.

4 Time-fractional KdV equation Solution

The time-FKdV equation represented by (18) can be solved using the VIM by the iteration correction functional (21) as follows:

Affecting from left by the fractional operator on (18) leads to

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= {}_0^R D_t{}^{\alpha-1} u(x,t)|_{t=0} \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \\ &- {}_0^R D_t{}^{1-\alpha}[A\ u(x,t)\frac{\partial}{\partial x}u(x,t) + B\ \frac{\partial^3}{\partial x^3}u(x,t)], \\ 0 &\leq \alpha \leq 1,\, t \in [0, T_0],\end{aligned} \tag{23}$$

where the following fractional derivative property is used [18, 19]

$$
\begin{aligned}
{}_a^R D_b{}^{\alpha}[\,{}_a^R D_b{}^{\beta} f(t)] &= {}_a^R D_b{}^{\alpha+\beta} f(t) - \sum_{j=1}^{k} {}_a^R D_b{}^{\beta-j} f(t)|_{t=a}\, \frac{(t-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)}, \\
k-1 &\leq \beta < k. \qquad (24)
\end{aligned}
$$

As $\alpha < 1$, the Riesz fractional derivative ${}_0^R D_t{}^{\alpha-1}$ is considered as Riesz fractional integral ${}_0^R I_t^{1-\alpha}$ that is defined by [18, 19]

$$
{}_0^R I_t{}^{\alpha} f(t) = \frac{1}{2}[{}_0 I_t{}^{\alpha} f(t) + {}_t I_b{}^{\alpha} f(t)] = \frac{1}{2}\frac{1}{\Gamma(\alpha)}\int_a^b d\tau |t-\tau|^{\alpha-1} f(\tau),\ \alpha > 0. \quad (25)
$$

where ${}_0 I\ {}_t{}^{\alpha} f(t)$ and ${}_t I_b{}^{\alpha} f(t)$ are the left and right Riemann-Liouvulle fractional integrals, respectively [18, 19].

The iterative correction functional of equation (23) is given as

$$
\begin{aligned}
u_{n+1}(x,t) &= u_n(x,t) + \int_0^t d\tau \lambda(\tau)\{\frac{\partial}{\partial \tau} u_n(x,\tau) \\
&- {}_0^R I_\tau^{1-\alpha} u_n(x,\tau)|_{\tau=0} \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \\
&+ {}_0^R D_\tau{}^{1-\alpha}[A\ \tilde{u}_n(x,\tau)\frac{\partial}{\partial x}\tilde{u}_n(x,\tau) + B\ \frac{\partial^3}{\partial x^3}\tilde{u}_n(x,\tau)]\}, \quad (26)
\end{aligned}
$$

where $n \geq 0$ and the function $\tilde{u}_n(x,t)$ is considered as a restricted variation function, i. e; $\delta\widetilde{u}_n(x,t) = 0$. The extreme of the variation of (26) using the restricted variation function leads to

$$
\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \int_0^t d\tau \lambda(\tau)\ \delta \frac{\partial}{\partial \tau} u_n(x,\tau) \\
&= \delta u_n(x,t) + \lambda(\tau)\ \delta u_n(x,t) - \int_0^t d\tau \frac{\partial}{\partial \tau}\lambda(\tau)\ \delta u_n(x,\tau) = 0.
\end{aligned}
$$

This relation leads to the stationary conditions $1+\lambda(t) = 0$ and $\frac{\partial}{\partial \tau}\lambda(\tau) = 0$, which leads to the Lagrangian multiplier as $\lambda(\tau) = -1$.

Therefore, the correction functional (32) is given by the form

$$
\begin{aligned}
u_{n+1}(x,t) &= u_n(x,t) - \int_0^t d\tau \{\frac{\partial}{\partial \tau} u_n(x,\tau) \\
&- {}_0^R I_\tau^{1-\alpha} u_n(x,\tau)|_{\tau=0} \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \\
&+ {}_0^R D_\tau{}^{1-\alpha}[A\ u_n(x,\tau)\frac{\partial}{\partial x} u_n(x,\tau) + B\ \frac{\partial^3}{\partial x^3} u_n(x,\tau)]\}, \quad (27)
\end{aligned}
$$

where $n \geq 0$.

In Physics, if t denotes the time-variable, the right Riemann-Liouville fractional derivative is interpreted as a future state of the process. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development [3]. Therefore, the right-derivative is used equal to zero in the following calculations.

The zero order correction of the solution can be taken as the initial value of the state variable, which is taken in this case as

$$u_0(x,t) = u(x,0) = A_0 \sec\mathrm{h}^2(cx). \tag{28}$$

where A_0 and c are constants.

Substituting this zero order approximation into (27) and using the definition of the fractional derivative (19) lead to the first order approximation as

$$\begin{aligned} u_1(x,t) \quad = \quad & A_0 \sec\mathrm{h}^2(cx) \\ & +2A_0 c \ \sinh(cx) \ \sec\mathrm{h}^3(cx)[4c^2 B \\ & +(A_0 A - 12c^2 B)\sec\mathrm{h}^2(cx)]\frac{t^{\ \alpha}}{\Gamma(\alpha+1)}. \end{aligned} \tag{29}$$

Substituting this equation into (27), using the definition (19) and the Maple package lead to the second order approximation in the form

$$\begin{aligned} u_2(x,t) \quad = \quad & A_0 \sec\mathrm{h}^2(cx) \\ & +A_0 c \ \sinh(cx) \sec\mathrm{h}^3(cx) \\ & *[4c^2 B + (A_0 A - 12c^2 B)\sec\mathrm{h}^2(cx)]\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ & +A_0 c^2 \sec\mathrm{h}^2(cx) \\ & *[16c^4 B^2 + 8c^2 B(5A_0 A - 63c^2 B)\sec\mathrm{h}^2(cx) \\ & +(3A_0^2 A^2 - 176A_0 c^2 AB + 168c^4 B^2)\sec\mathrm{h}^4(cx) \\ & -\frac{7}{2}(A_0^2 A^2 - 42A_0 c^2 AB + 360c^4 B^2)\sec\mathrm{h}^6(cx)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ & +A_0^2 c^3 \sinh(cx) \ \sec\mathrm{h}^5(cx) \\ & *\{4c^2 B[4c^2 B + 3(A_0 A - 14c^2 B)\sec\mathrm{h}^2(cx)] \\ & +2(A_0^2 A^2 - 32A_0 c^2 AB + 240c^4 B^2)\sec\mathrm{h}^4(cx) \\ & -\frac{5}{2}(A_0^2 A^2 - 24A_0 c^2 AB + 144c^4 B^2)\sec\mathrm{h}^6(cx)\} \\ & *\frac{\Gamma(2\alpha+1)}{[\Gamma(\alpha+1)]^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \end{aligned} \tag{30}$$

The higher order approximations can be calculated using the Maple or the Mathematica package to the appropriate order where the infinite approximation leads to the exact solution.

5 Results and calculations

The calculations are carried out for the solution of the time-FKdV equation using the VIM for different values of the equation parameters of nonlinearity (A) and dispersion (B). The initial value of the solution for all cases is taken as $A_0 \operatorname{sech}(cx)$ where the amplitude A_0 is taken equal to unity and the constant (c) is used with different values as B/A, $B/(4A)$, $B/(8A)$ or $B/(12A)$. The solution is calculated for different values of the fractional order α ($\alpha = 3/4$, $\alpha = 1/2$, $\alpha = 1/3$ and $\alpha = 1/4$).

The 3-dimensional representation of the solution of the time-FKdV equation with space x and time t for constant $c = B/(4A)$ and $A = 1$, $B = 1$ for different values of the fractional order (α) is given in Fig (1). While in Fig (2), the 3-dimensional representation of the solution of time-FKdV equation with $A = 1$, $B = 1$ is given for the fractional order $\alpha = 1/2$ for different values of the constant $c = B/A$, $c = B/(4A)$, $c = B/(8A)$ and $c = B/(12A)$.

Figure (3) represents the solution of the time-FKdV equation for fractional order $\alpha = 1/2$ with $c = B/(4A)$ using ($A = 1$, $B = 1$), ($A = 1$, $B = 3$), ($A = 6$, $B = 1$) and ($A = 6$, $B = 3$).

In Fig (4), the solution of the time-FKdV equation $u(x, t)$, for ($A = 1$, $B = 1$) with $c = B/(4A)$, is calculated for $\alpha = 1/2$. These calculations are represented as 2-dimensional figure against the space x for different values of time t.

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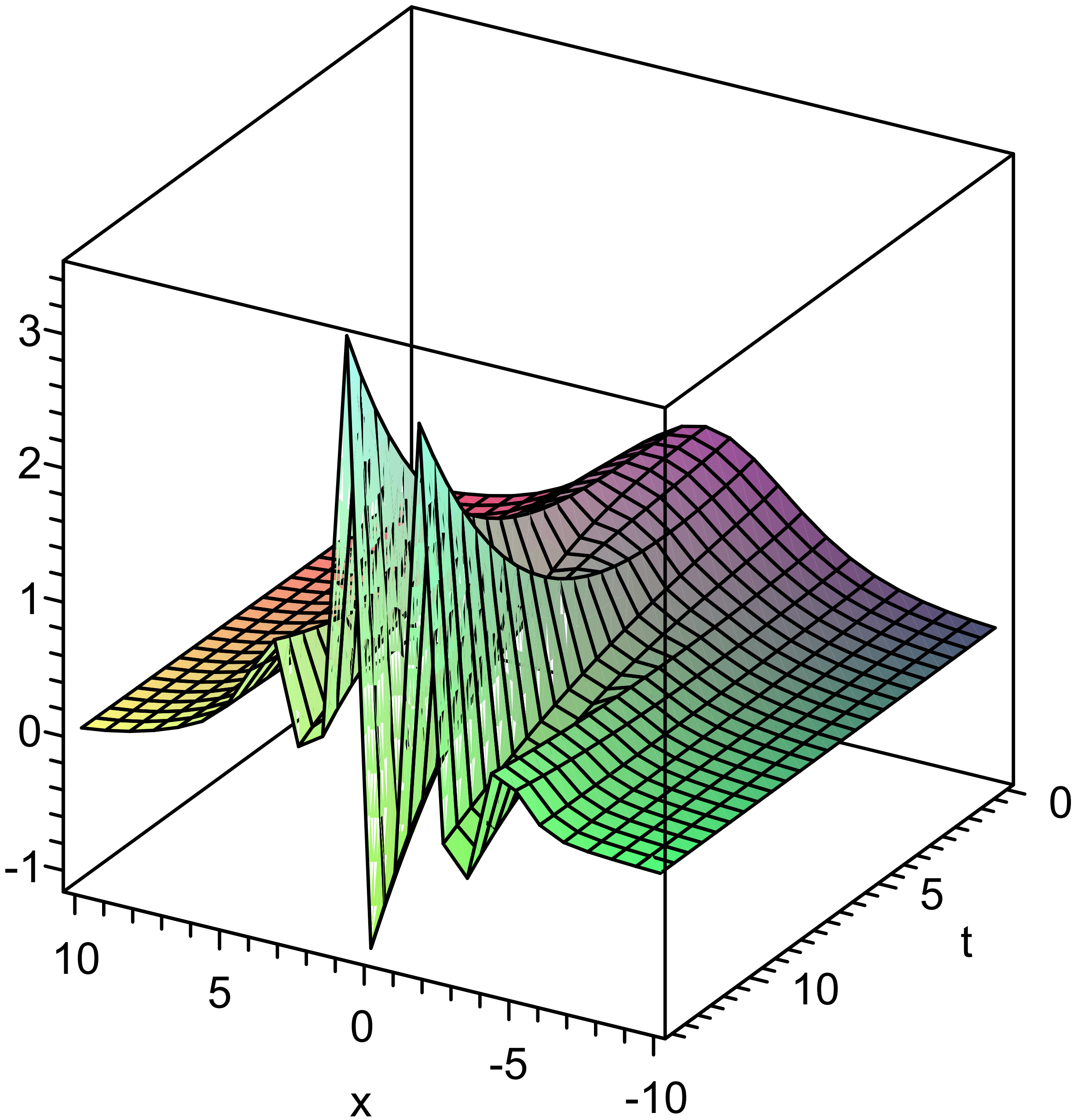
Figure Captions

Fig. 1: The distribution function $u(x,t)$ with $A = 1$, $B = 1$, initial condition $u_0 = \operatorname{sech}^2(cx)$ and $c = B/(4A)$ for different values of the fractional order (α).

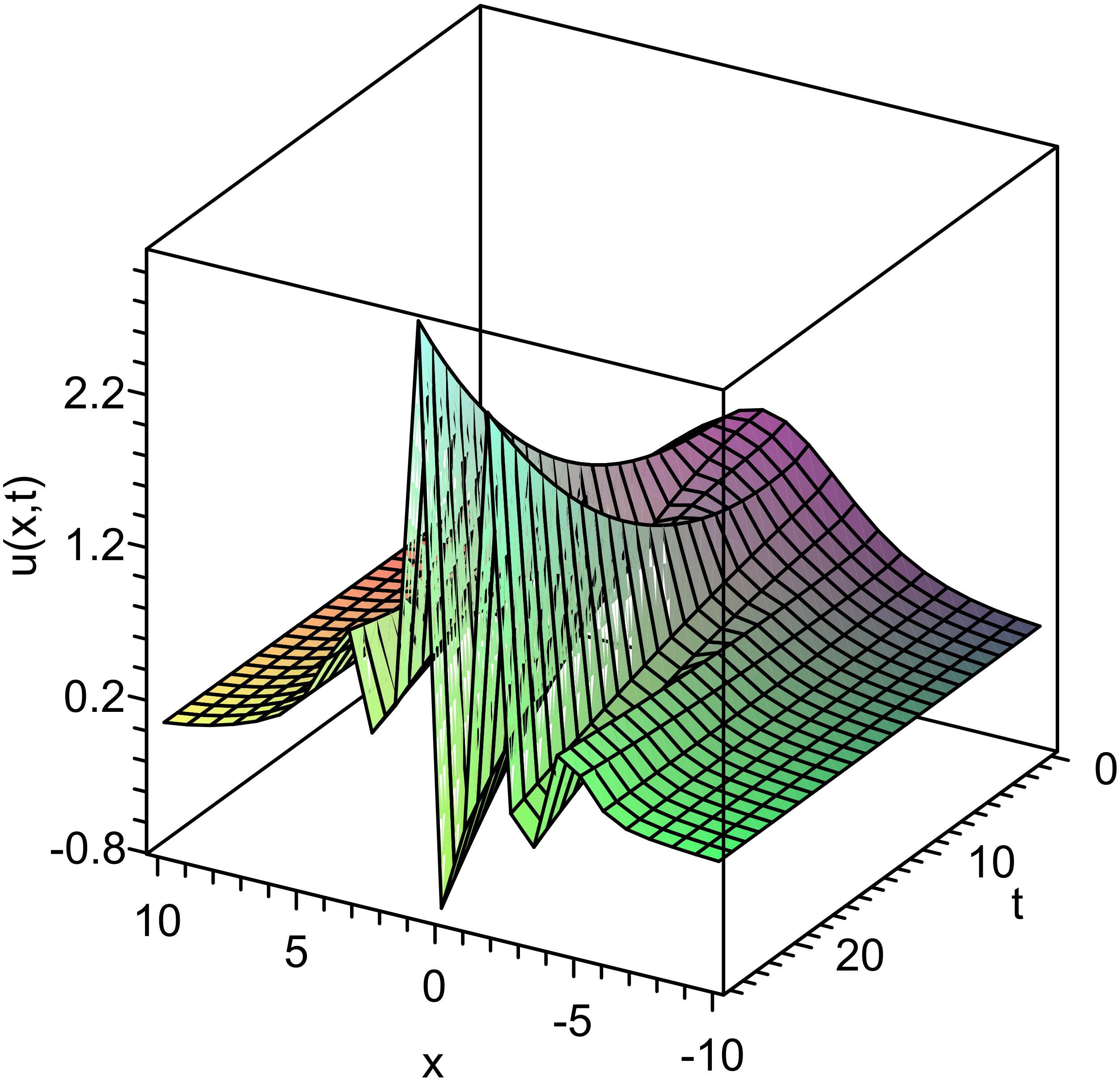
Fig. 2: The distribution function $u(x,t)$ with $A = 1$, $B = 1$, initial condition $u_0 = \operatorname{sech}^2(cx)$ and fractional order $\alpha = 1/2$ for different values of the constant (c).

Fig. 3: The distribution function $u(x,t)$ with fractional order $\alpha = 1/2$, initial condition $u_0 = \operatorname{sech}^2(cx)$ and $c = B/(4A)$ for different values of A and B.

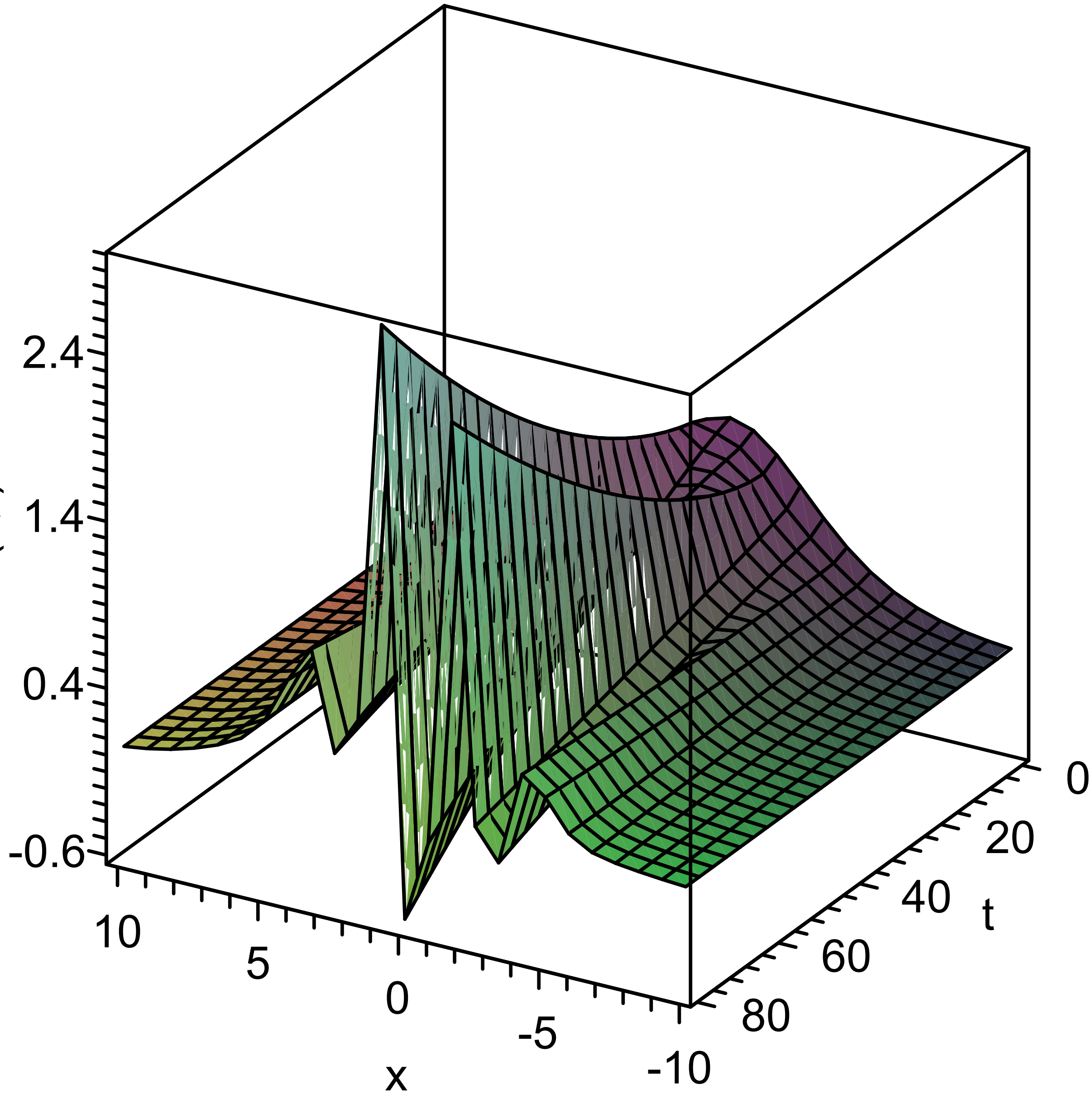
Fig. 4: The distribution function $u(x,t)$ as a function of space (x) for different values of time t with $A = 1$, $B = 1$, initial condition $u_0 = \operatorname{sech}^2(cx)$, $c = B/(4A)$ and $\alpha = 1/2$.



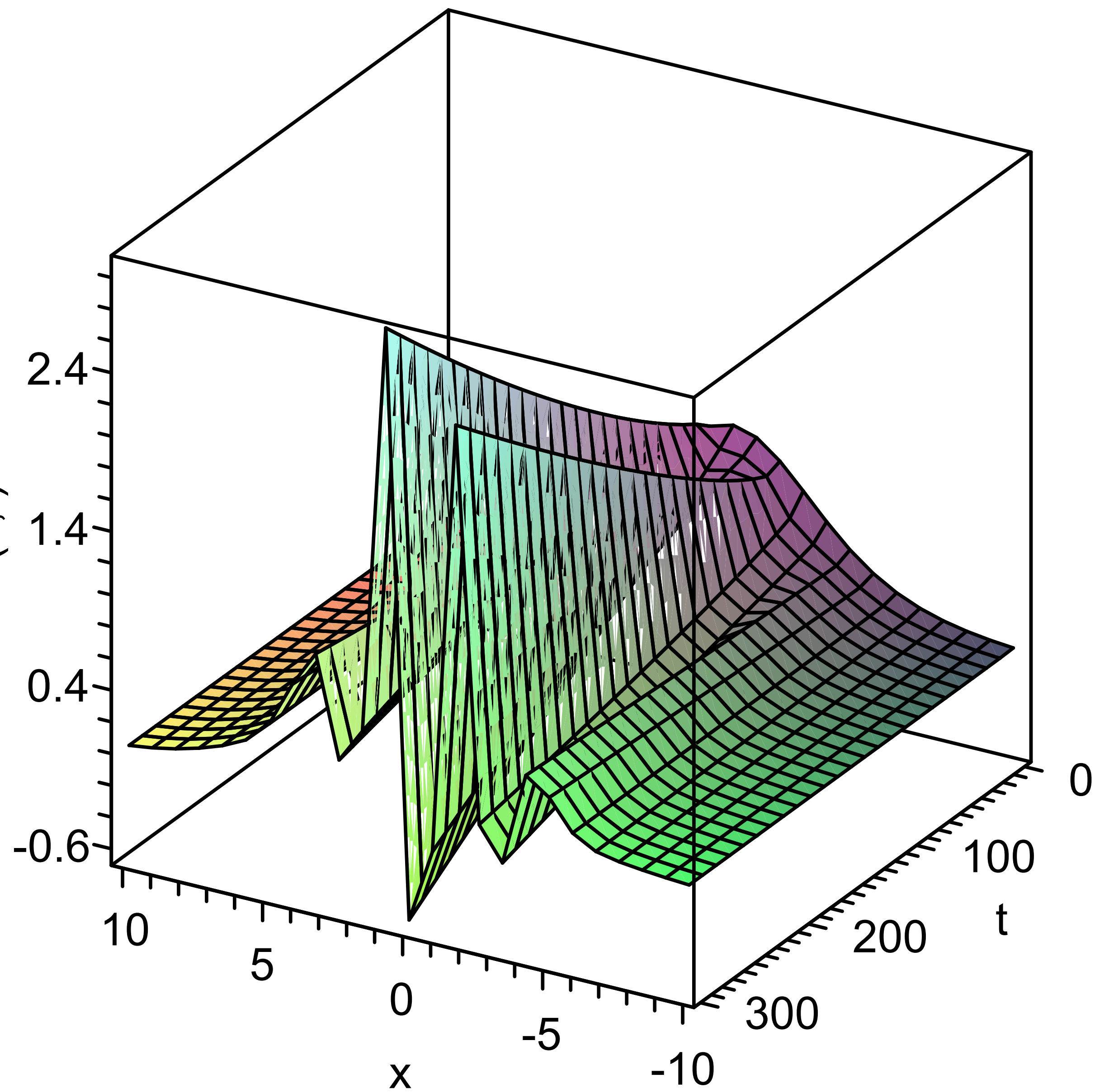
u(x,t)
3
2
1
0
-1
10
5
0
-5
-10
x
0
5
t
10

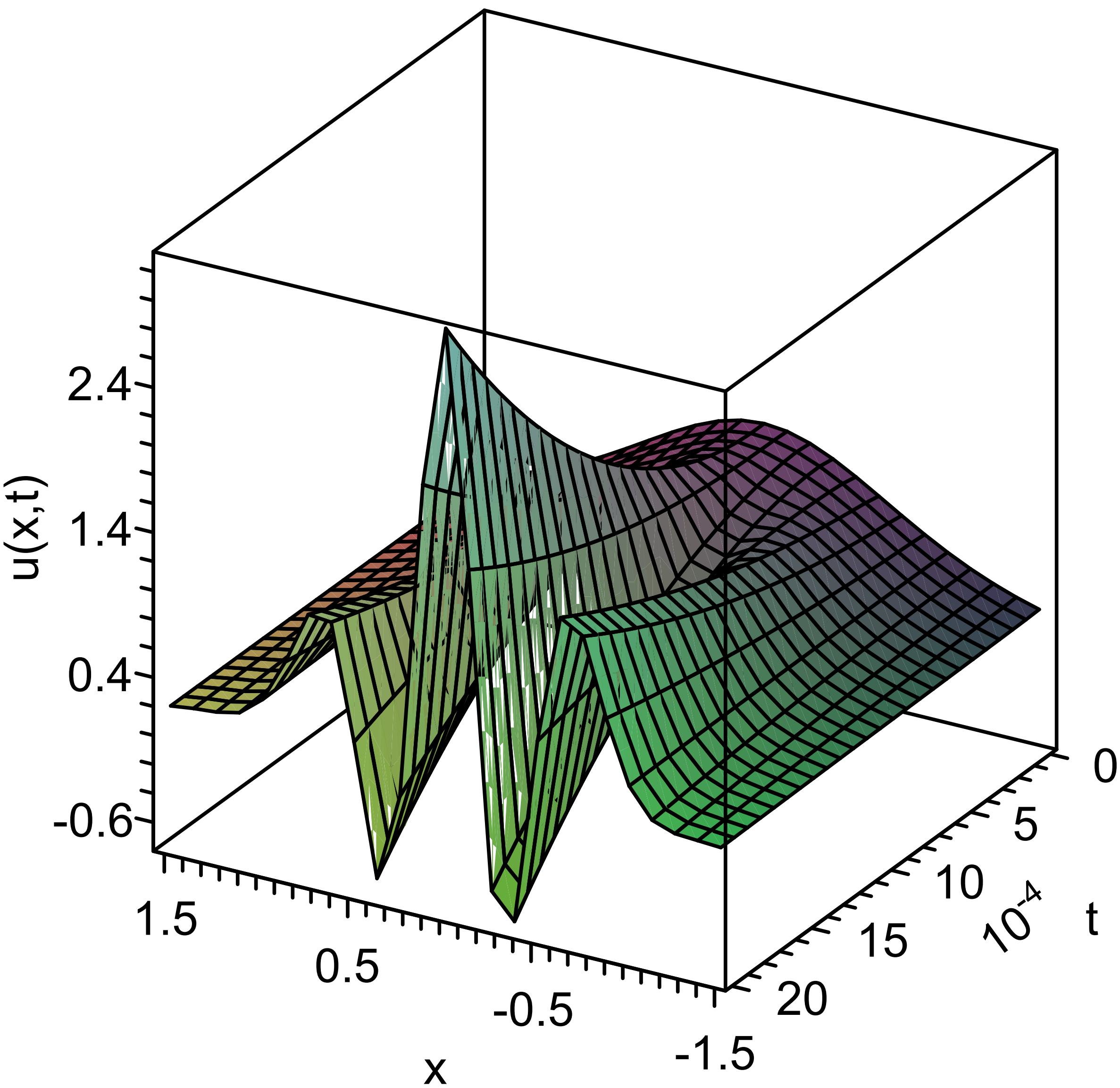


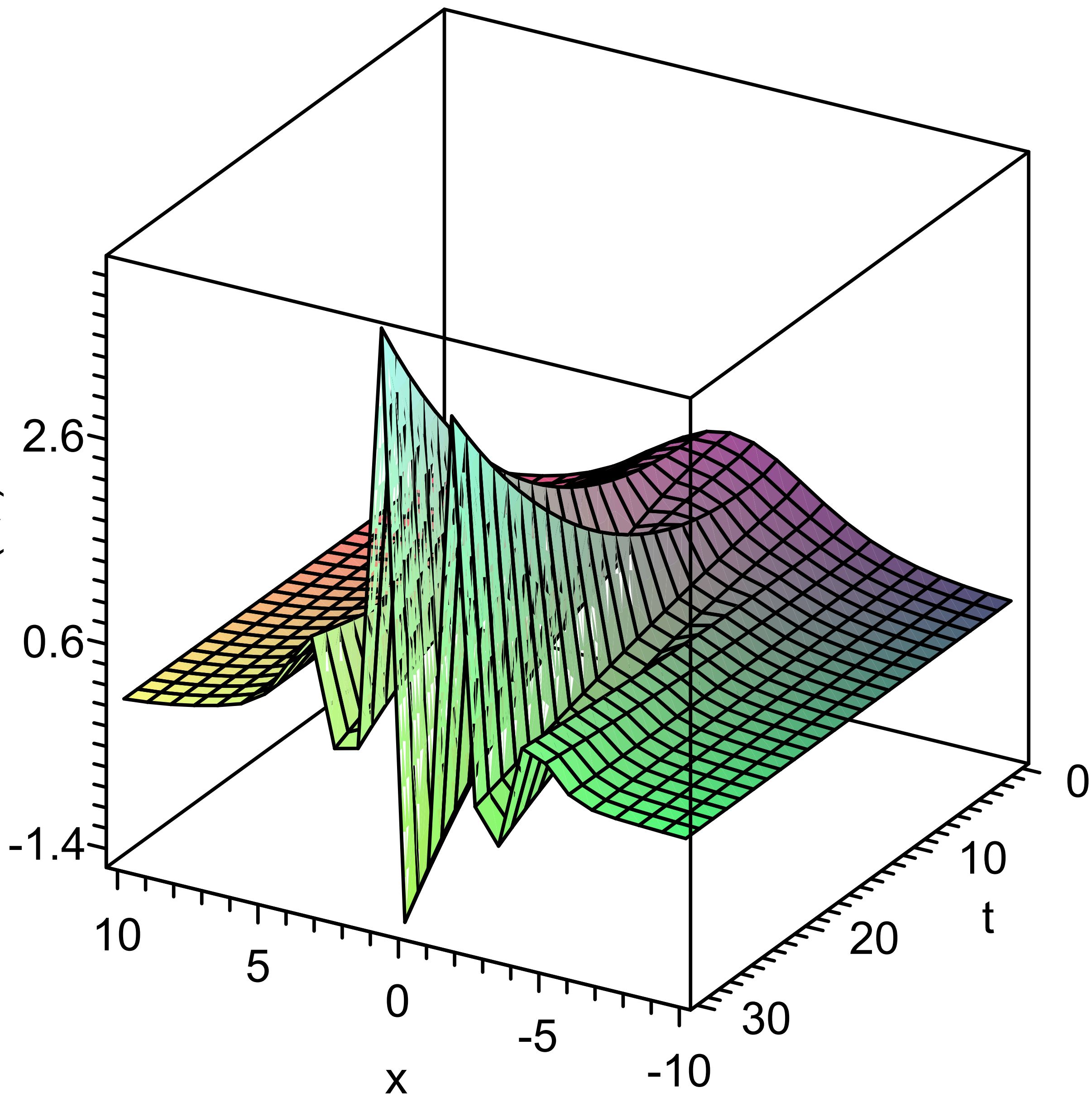
2.2
1.2
0.2
-0.8
u(x,t)
10
5
0
-5
-10
x
0
10
20
t



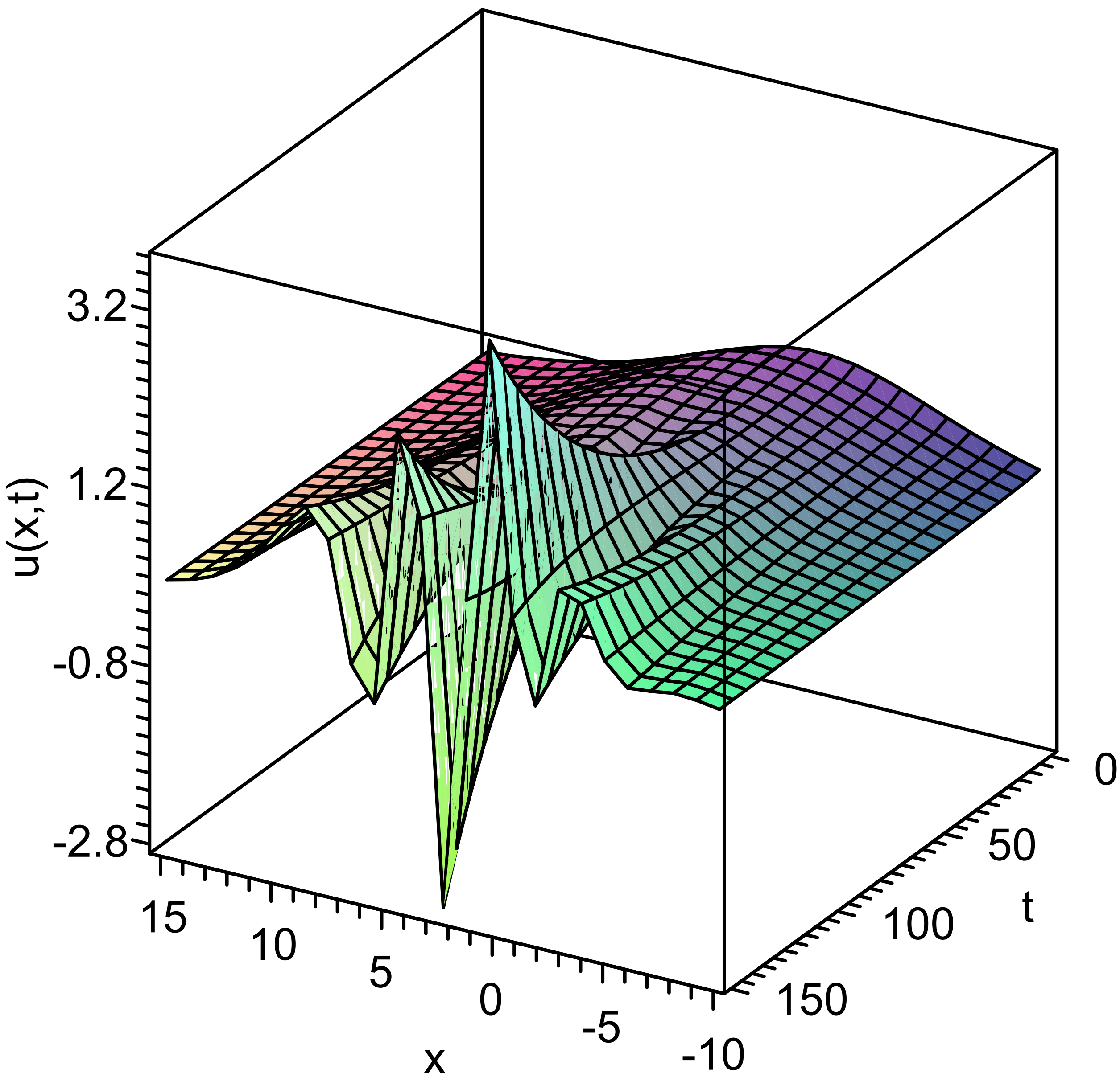
2.4
1.4
0.4
-0.6
u(x,t)
10
5
0
-5
-10
x
0
20
40
60
80
t


u(x,t)
2.4
1.4
0.4
-0.6
10
5
0
-5
-10
x
0
100
200
300
t

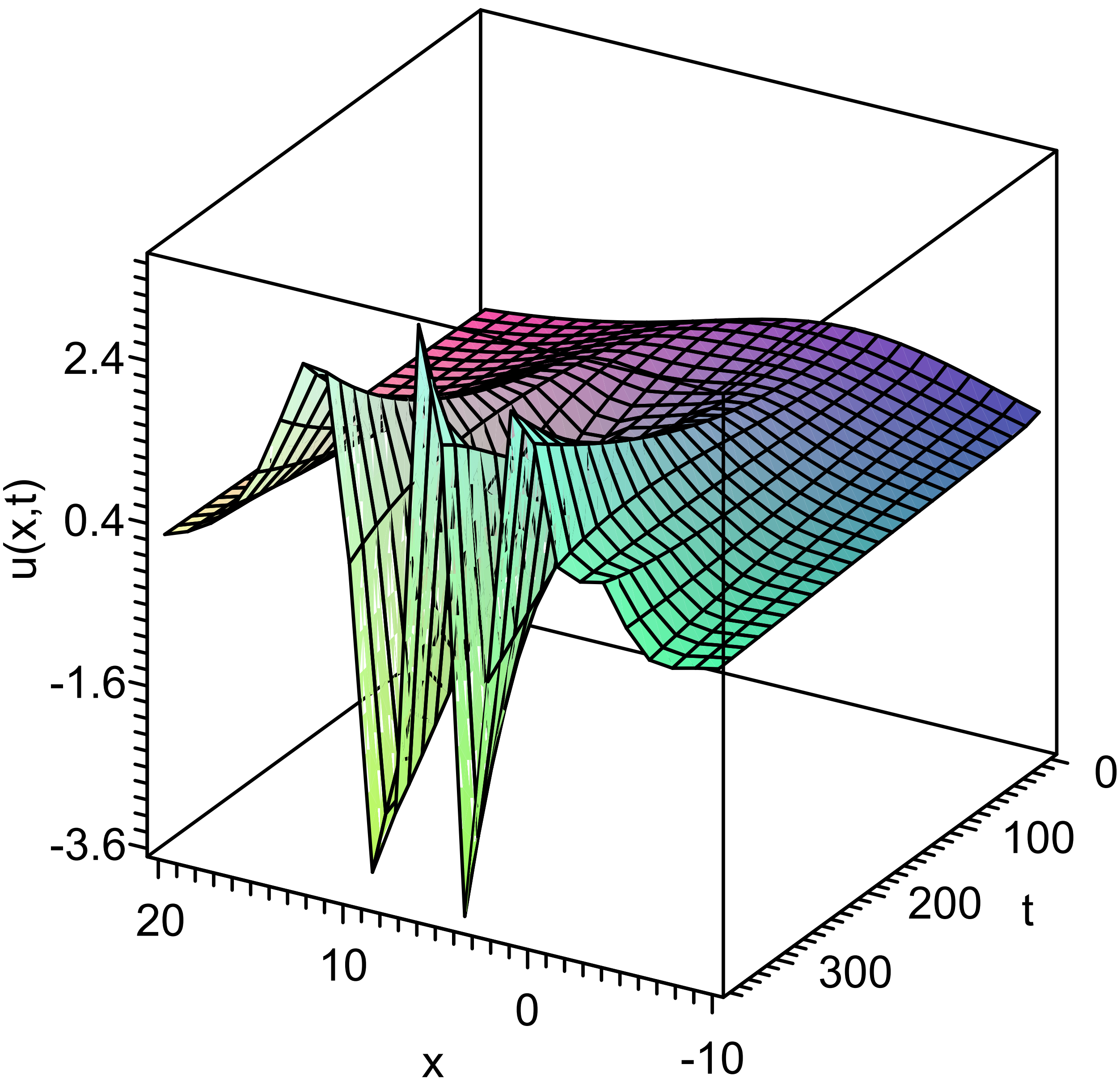

u(x,t)
2.4
1.4
0.4
-0.6
1.5
0.5
-0.5
-1.5
x
0
5
10
15
20
10^{-4}
t



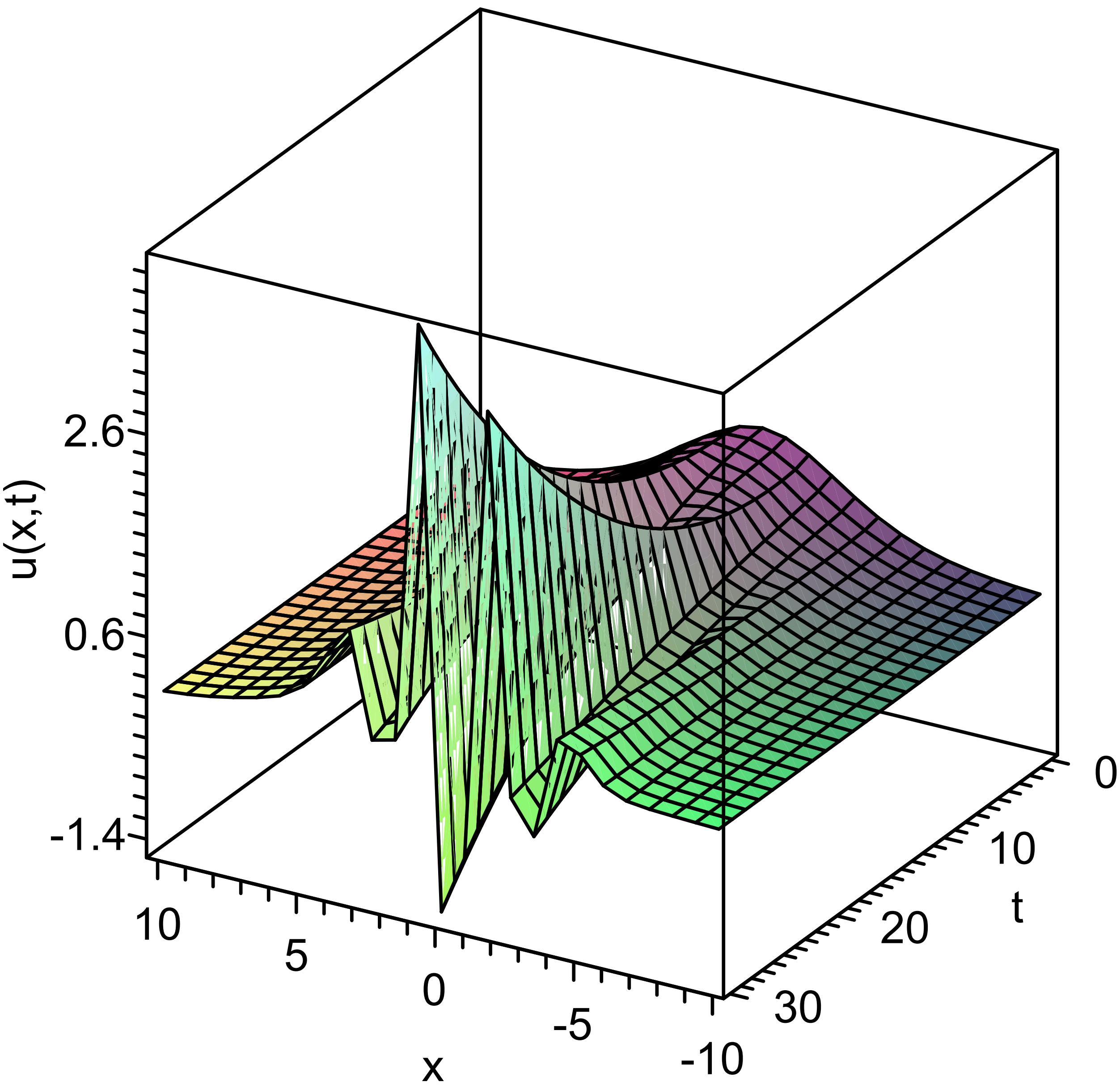
u(x,t)
2.6
0.6
-1.4
10
5
0
-5
-10
x
0
10
20
30
t



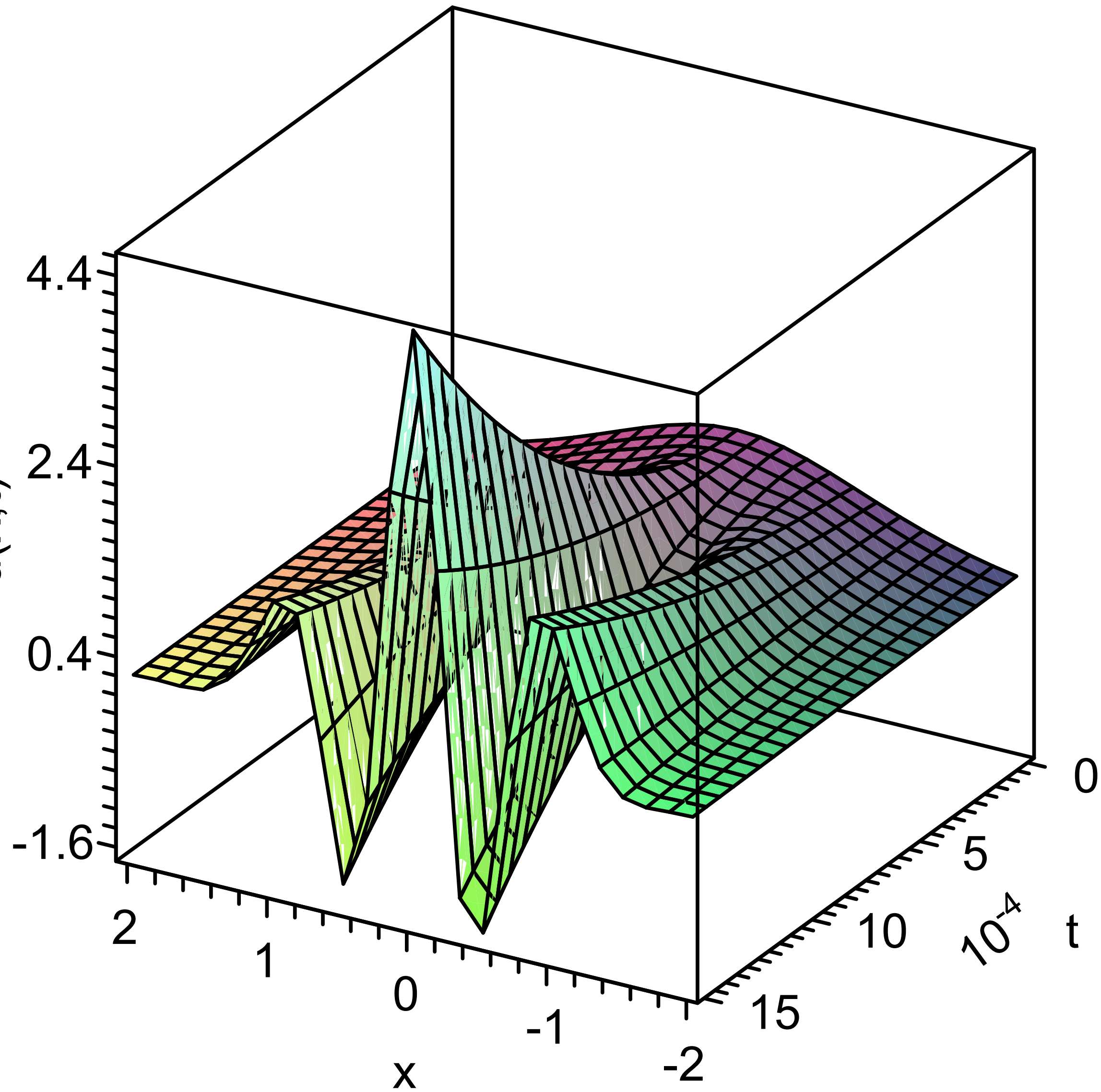
3.2
1.2
-0.8
-2.8
u(x,t)
15
10
5
0
-5
-10
x
0
50
100
150
t



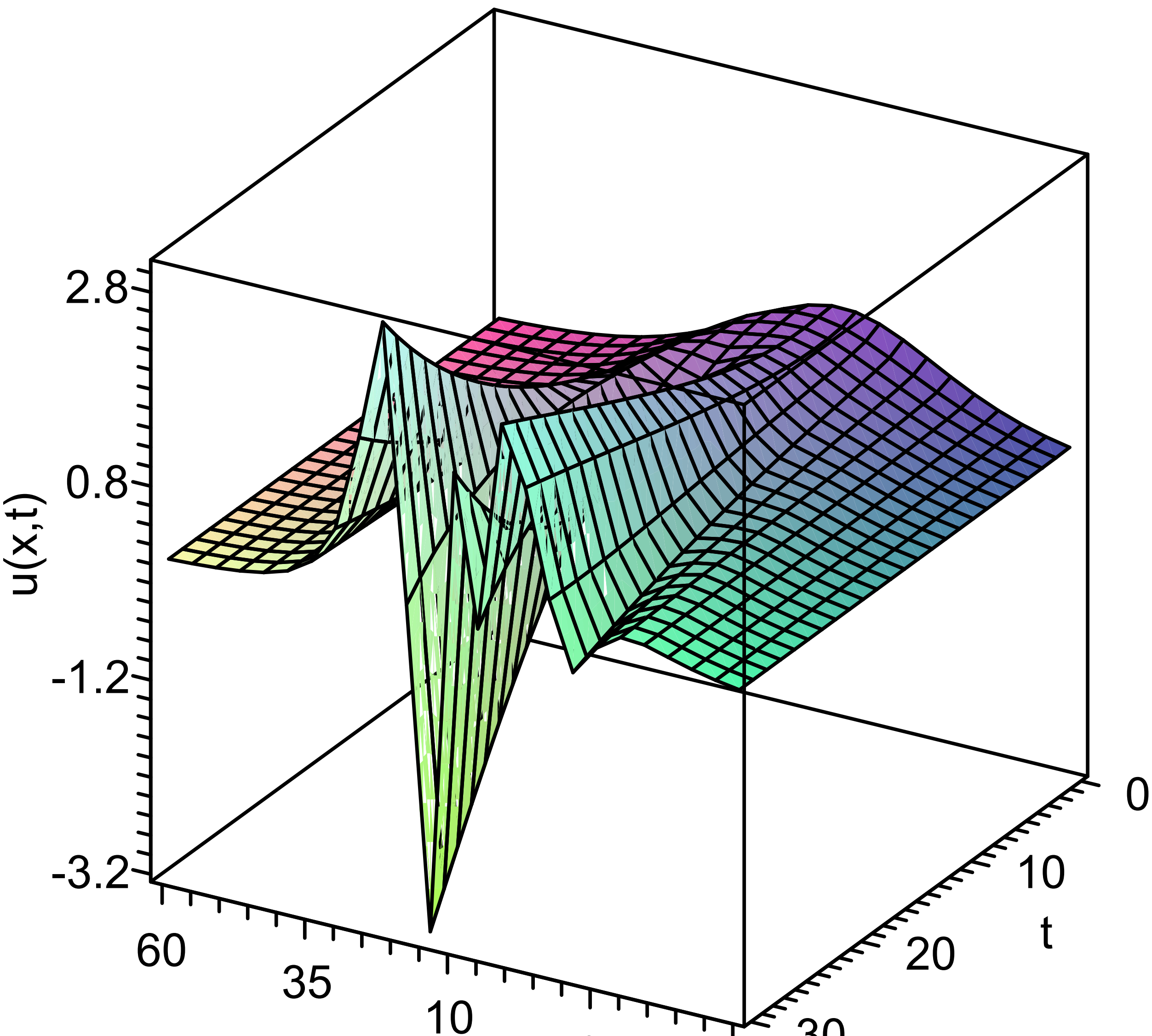
u(x,t)
2.4
0.4
-1.6
-3.6
20
10
0
-10
x
0
100
200
300
t



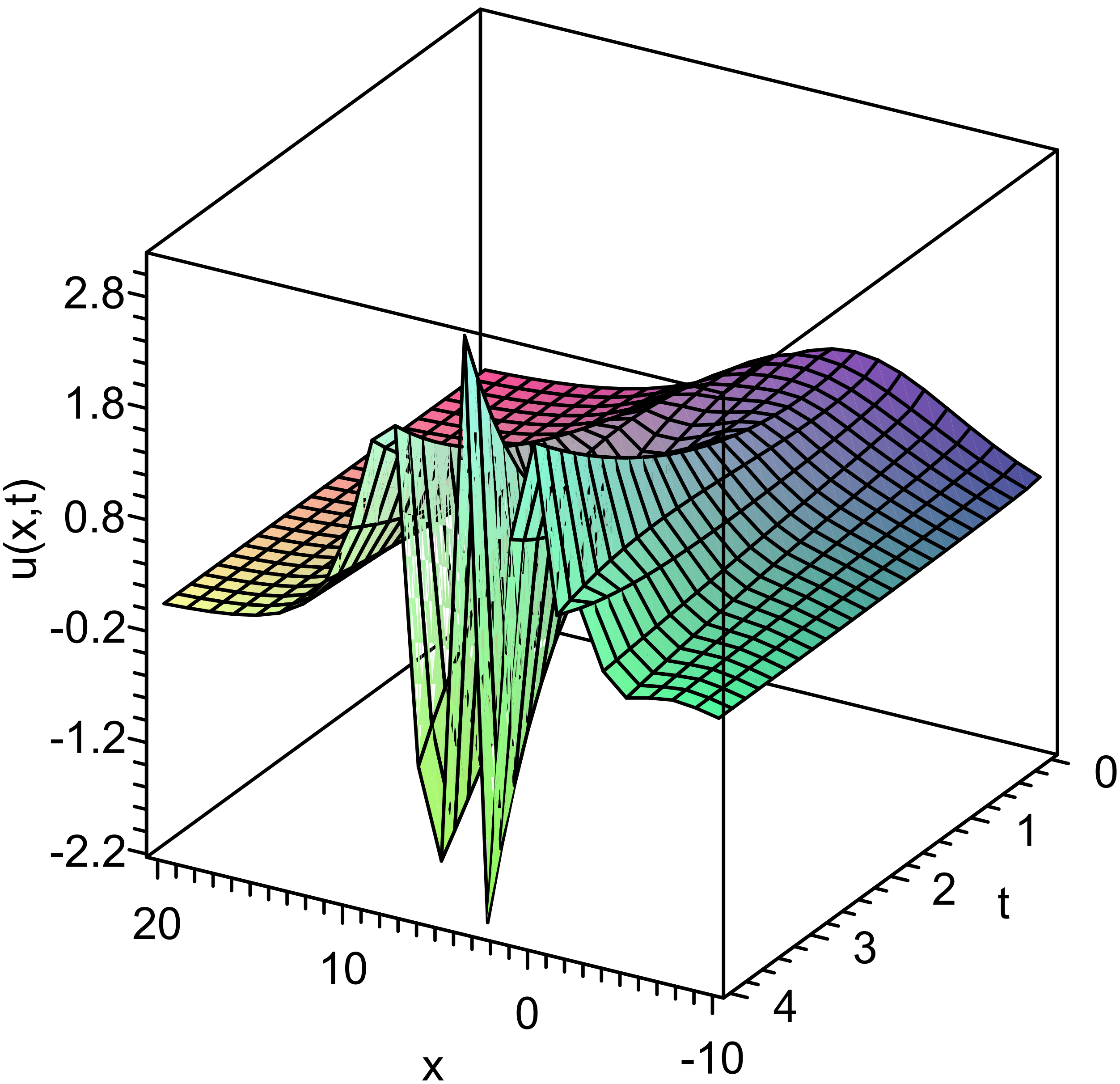
u(x,t)
2.6
0.6
-1.4
10
5
0
-5
-10
x
0
10
20
30
t



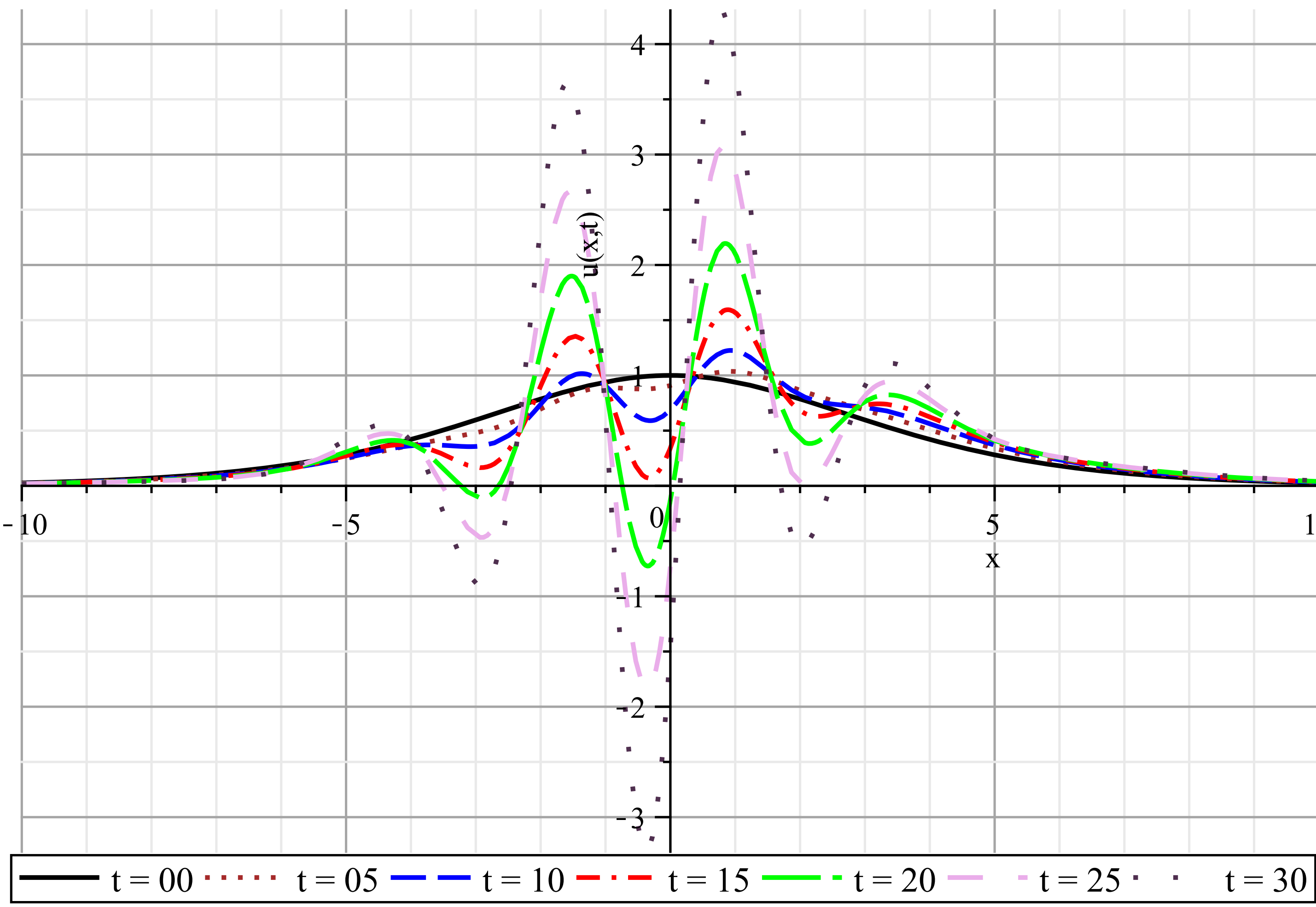
4.4
2.4
0.4
-1.6
u(x,t)
2
1
0
-1
-2
x
0
5
10
15
10^{-4}
t



2.8
0.8
-1.2
-3.2
u(x,t)
60
35
10
-15
-40
x
0
10
20
30
t



2.8
1.8
0.8
-0.2
-1.2
-2.2
u(x,t)
20
10
0
-10
x
0
1
2
3
4
t



4
3
2
1
u(x,t)
-1
-2
-3
-10
-5
0
5
10
x
t = 00
t = 05
t = 10
t = 15
t = 20
t = 25
t = 30